Hierarchy of piecewise non-linear maps with non-ergodicity behavior

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Abstract

We study the dynamics of hierarchy of piecewise maps generated by one-parameter families of trigonometric chaotic maps and one-parameter families of elliptic chaotic maps of **cn** and **sn** types, in detail. We calculate the Lyapunov exponent and Kolmogorov-Sinai entropy of the these maps with respect to control parameter. Non-ergodicity of these piecewise maps is proven analytically and investigated numerically . The invariant measure of these maps which are not equal to one or zero, appears to be characteristic of non-ergodicity behavior. A quantity of interest is the Kolmogorov-Sinai entropy, where for these maps are smaller than the sum of positive Lyapunov exponents and it confirms the non-ergodicity of the maps.

Keywords: Non-ergodicity, piecewise maps, chaos, Lyapunov exponent, Kolmogorov-Sinai entropy.

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1 Introduction

Ergodicity theory is a branch of dynamic systems dealing with questions of average. Sometimes ergodic theory can make long-term predictions about the average behavior from initial data with limited accuracy even for chaotic systems. The ergodic theory of chaos has been studied in detail by Eckmann and Ruelle [1, 2] and others, while non-ergodic mathematical models scarcely exist. It is shown that simplest one-dimensional dynamic systems satisfying the indecomposability assumption (and even the assumption of topological transitivity) may be non-ergodic, which shows that restrictions of this type are not quite reasonable in the context of general dynamical systems [3]. Also, ergodicity has not been proven for some systems and sometimes we have to face the problem of non-ergodicity of many chemical and physical systems especially in solid systems (for example see [4, 5, 6, 7]). It is shown that, candidate solid state systems with extremely slow dynamics often have impurities remaining from their synthesis so there is always a suspicion that non-ergodicity comes from disorder [8]. Also, there are specific examples of arbitrary small perturbation to ergodic systems that are not ergodic [9, 10, 11]. The study of origins of the non-ergodicity and slow dynamics of polymer gels [12] and effects of temperature and swelling on chain dynamics in sol-gel transition [13, 14] and also non-ergodicity transition in colloidal gelation [15] are some examples of research activity concerning non-ergodicity.

The aim of present paper is twofold: To introduce the piecewise map with its invariant measure and to clarify its non-ergodic behavior. We present the hierarchy of one and many-parameter families of elliptic chaotic maps of **cn** and **sn** types which can generate non-ergodic behavior. In our definition of a piecewise map, we assume that it consist of components which can describe fixed point and chaotic behavior accordance with various values of parameter of the map. In particular, we argue that this map satisfy the non-ergodic assumption. The paper is organized as follows: The next section of the paper will note the competing definitions of piecewise non-linear maps with complete boundary condition associated with the

one parameter families of chaotic maps and one-parameter families of elliptic chaotic maps of **cn** and **sn** types. In Section 3, we present the invariant measure of piecewise maps and in Section 4 the Kolmogorov-Sinai (KS) entropy and Lyapunov exponent (LE) of the map is studied. In Section 5, we review the ergodic theory and in Section 6, we give our results consist the KS-entropy and LE of the map and we shall explain how this piecewise maps behave non-ergodic. In addition there will be a concluding section and two appendices.

2 Piecewise non-linear maps

2.1 One-parameter families of trigonometric chaotic maps

We first review the one-parameter chaotic maps which are used to construct the piecewise map. The one-parameter chaotic maps [16] are defined as the ratio of polynomials of degree **N**:

$$\begin{aligned}\phi_N^{(1)}(x,\alpha) &= \frac{\alpha^2(1+(-1)^N \quad {}_2F_1(-N,N,\frac{1}{2},x))}{(\alpha^2+1)+(\alpha^2-1)(-1)^N \quad {}_2F_1(-N,N,\frac{1}{2},x)} \\ &= \frac{\alpha^2(T_N(\sqrt{x}))^2}{1+(\alpha^2-1)(T_N(\sqrt{x})^2)} \\ \phi_N^{(2)}(x,\alpha) &= \frac{\alpha^2(1-(-1)^N \quad {}_2F_1(-N,N,\frac{1}{2},(1-x)))}{(\alpha^2+1)-(\alpha^2-1)(-1)^N \quad {}_2F_1(-N,N,\frac{1}{2},(1-x))} \\ &= \frac{\alpha^2(U_N(\sqrt{(1-x)}))^2}{1+(\alpha^2-1)(U_N(\sqrt{(1-x)})^2)}\end{aligned}$$

where **N** is an integer greater than one. Also,

$${}_2F_1(-N,N,\frac{1}{2},x) = (-1)^N\cos(2N\arccos\sqrt{x}) = (-1)^N T_{2N}(\sqrt{x})$$

is the hypergeometric polynomials of degree N and $T_N(U_n(x))$ are Chebyshev polynomials of type I (type II), respectively. The conjugate maps of the one-parameter families of chaotic

maps which are used in derivation of their invariant measure and calculation of their KS-entropy are defined as:

$$\tilde{\phi}_N^{(1)}(x,\alpha) = h \circ \phi_N^{(1)}(x,\alpha) \circ h^{-1} = \frac{1}{\alpha^2}\tan^2(N \arctan \sqrt{x})$$

$$\tilde{\phi}_N^{(2)}(x,\alpha) = h \circ \phi_N^{(2)}(x,\alpha) \circ h^{-1} = \frac{1}{\alpha^2}\cot^2(N \arctan \frac{1}{\sqrt{x}})$$

Conjugacy means that invertible map $h(x) = \frac{1-x}{x}$ maps $I = [0,1]$ into $[0,\infty)$. To define the piecewise maps constructed from one-parameter chaotic map, we need to take into account boundary condition, namely we have to choose the states on the phase space. Here we present examples of these types which have been considered in the present paper:

$$\phi_2^{(1)} = \frac{\alpha^2(2x-1)^2}{4x(1-x)+\alpha^2(2x-1)^2}$$

$$\phi_2^{(2)} = \frac{4\alpha^2 x(1-x)}{1+4(\alpha^2-1)x(1-x)}$$

$$\phi_3^{(1)} = \phi_3^{(2)} = \frac{\alpha^2 x(4x-3)^2}{\alpha^2 x(4x-3)^2+(1-x)(4x-1)^2}$$

Now, We introduce hierarchy of piecewise maps generated by one-parameter families of trigonometric chaotic maps. For piecewise map $\phi_N^{(1)}$ with even $\mathbf{N}$, we have:

$$\phi_N^{(1)}(x,\alpha) = \begin{cases} \phi_N^{(1)}(x,\alpha_1) & \alpha_1 \in [N,\infty) \\ \phi_N^{(1)}(x,\alpha_2) & \alpha_2 \in [0,N] \end{cases}$$

The range of the parameters α_1 and α_2 in the maps $\phi_N^{(1)}(x,\alpha_1)$ and $\phi_N^{(1)}(x,\alpha_2)$ are chosen to grantee, respectively, chaotic behavior and two fixed points at $x = 0$ and $x = 1$. Figure 1 shows the plot of $\phi_2^{(1)}(x,\alpha)$ for $\alpha_1 = 3$ and $\alpha_2 = 1$. In $\phi_N^{(1)}(x,\alpha_1)$ and $\phi_N^{(1)}(x,\alpha_2)$, $\mathbf{x}$ is limited respectively to: $\acute{x} \in [0.152, 0.848]$ and $\ddot{x} \in [0, 0.352] \cup [0.647, 1]$. For given $y_0 = 0.5$ the $\acute{x}$ are the roots of $\phi_N^{(1)}(x,\alpha_1) = y_0$ and similarly the $\ddot{x}$ are the roots of $\phi_N^{(1)}(x,\alpha_2) = y_0$. For piecewise map $\phi_N^{(2)}(x,\alpha)$ with even $\mathbf{N}$, we have:

$$\phi_N^{(2)}(x,\alpha) = \begin{cases} \phi_N^{(2)}(x,\alpha_1) & \alpha_1 \in [0, \frac{1}{N}] \\ \phi_N^{(2)}(x,\alpha_2) & \alpha_2 \in [\frac{1}{N}, \infty) \end{cases}$$

The range of the parameters α_1 and α_2 in the maps $\phi_N^{(2)}(x,\alpha_1)$ and $\phi_N^{(2)}(x,\alpha_2)$ are chosen to grantee, respectively, two fixed points at $x = 0$ and $x = 1$ and chaotic behavior. Figure 2 shows the plot of $\phi_2^{(2)}(x,\alpha)$ for $\alpha_1 = 0.25$ and $\alpha_2 = 0.75$. In $\phi_N^{(2)}(x,\alpha_1)$ and $\phi_N^{(2)}(x,\alpha_2)$, **x** is limited respectively to: $\acute{x} \in [0, 0.378] \cup [0.621, 1]$ and $\ddot{x} \in [0.2, 0.8]$. For given $y_0 = 0$ the $\acute{x}$ are the roots of $\phi_N^{(2)}(x,\alpha_1) = y_0$ and similarly the $\ddot{x}$ are the roots of $\phi_N^{(2)}(x,\alpha_2) = y_0$.
For piecewise map $\phi_N^{(1,2)}$ with odd **N**, we have:

$$\phi_N^{(1,2)}(x,\alpha) = \begin{cases} \phi_N^{(1,2)}(x,\alpha_1) & \alpha_1 \in [\frac{1}{N}, N] \\ \phi_N^{(1,2)}(x,\alpha_2) & \alpha_2 \in [0, \frac{1}{N}] \cup [N, \infty) \end{cases}$$

This map has chaotic behavior for α_2 and has a fixed point in $x = 0$ for α_1. Figure 3 shows the plot of $\phi_3^{(1,2)}(x,\alpha)$ for $\alpha_1 = 1.5$ and $\alpha_2 = 0.2$. In $\phi_N^{(1,2)}(x,\alpha_1)$ and $\phi_N^{(1,2)}(x,\alpha_2)$, **x** is limited respectively to: $\acute{x} \in [0, 0.2] \cup [0.315, 1]$ and $\ddot{x} \in [0.086, 0.453] \cup [0.947, 1]$. For given $y_0 = 0.5$ the $\acute{x}_i$ are the roots of $\phi_N^{(2)}(x,\alpha_1) = y_0$ and similarly the $\ddot{x}_i$ are the roots of $\phi_N^{(2)}(x,\alpha_2) = y_0$.

2.2 One-parameter families of elliptic chaotic maps of cn and sn types

Here we first review a hierarchy of one-parameter families of elliptic of **cn** and **sn** types that have been used for constructing the piecewise maps with non-ergodic behavior. These kinds of maps are defined as the ratios of Jacobian elliptic functions of **cn** and **sn** types through the following equation [17]:

$$\phi_N^{(1)}(x,\alpha) = \frac{\alpha^2 (cn(Ncn^{-1}(\sqrt{x})))^2}{1 + (\alpha^2 - 1)(cn(Ncn^{-1}(\sqrt{x})))^2}$$

$$\phi_N^{(2)}(x,\alpha) = \frac{\alpha^2(sn(Nsn^{-1}(\sqrt{x})))^2}{1+(\alpha^2-1)(sn(Nsn^{-1}(\sqrt{x})))^2}$$

where α is control parameter. For $N=2$ we have:

$$\phi_2^{(1)}(x,\alpha) = \frac{\alpha^2((1-k^2)(2x-1)+k^2x^2)^2}{(1-k^2+2k^2x-k^2x^2)^2+(\alpha^2-1)((1-k^2)(2x-1)+k^2x^2)^2}$$

$$\phi_2^{(2)}(x,\alpha) = \frac{4\alpha^2x(1-k^2x)(1-x)}{(1-k^2x^2)^2+4x(1-x)(\alpha^2-1)(1-k^2x)}$$

It has been proved [17] that for small values of the parameter $\mathbf{K}$ of the elliptic function, these maps are topologically conjugate to the one parameter families of chaotic maps. Similar to the introduced piecewise in previous section (2.1), the piecewise of elliptic maps can be introduced. For an example for piecewise elliptic map $\phi_2^{(2)}(x,\alpha)$, we have:

$$\phi_2^{(2)}(x,\alpha) = \begin{cases} \phi_2^{(2)}(x,\alpha_1) & \alpha_1 \in [0, \frac{1}{N}] \\ \phi_2^{(2)}(x,\alpha_2) & \alpha_2 \in [\frac{1}{N}, \infty) \end{cases}$$

The range of the parameters α_1 and α_2 in the maps $\phi_2^{(2)}(x,\alpha_1)$ and $\phi_2^{(2)}(x,\alpha_2)$ are chosen to grantee, respectively, two fixed points at $x=0$ and $x=1$ and chaotic behavior. Figure 4 shows the plot of the elliptic map $\phi_2^{(2)}(x,\alpha)$ for $\alpha_1=0.5$ and $\alpha_2=2.5$. In $\phi_2^{(2)}(x,\alpha_1)$ and $\phi_2^{(2)}(x,\alpha_2)$, $\mathbf{x}$ is limited respectively to: $\acute{x} \in [0,0.28] \cup [0.72,1]$ and $\ddot{x} \in [0.027, 0.973]$. For given $y_0=0.5$ the $\acute{x}$ are the roots of $\phi_2^{(2)}(x,\alpha_1)=y_0$ and similarly the $\ddot{x}_i$ are the roots of $\phi_2^{(2)}(x,\alpha_2)=y_0$.

3 Invariant measure

Invariant measure or SRB measure is supported on an attractor and describes the statistical of long-time behavior of the orbits with respect to Lebesgue measure. For invariant measure of $\phi_N^{(i)}$ map $(i=1,2)$ satisfying the Frobenius-Perron (FP) operator [18], we have:

$$\mu(y) = \int_0^1 \delta(y-\phi_N^{(i)}(x,\alpha))\mu(x)dx$$

which is equivalent to:

$$\mu(y) = \sum_{x \in \phi_N^{-1(i)}(y,\alpha)} \mu(x) \frac{dx}{dy} \tag{3-1}$$

For chaotic part of the piecewise map, i.e.; $y \in [0, y_0]$ for $\phi_2^{(1)}(x,\alpha)$ and $y \in [y_0, 1]$ for both $\phi_3^{(1,2)}(x,\alpha)$ and $\phi_2^{(2)}(x,\alpha)$, the invariant measure $\mu(x,\beta)$ is defined as:

$$\frac{1}{\pi} \frac{\sqrt{\beta}}{\sqrt{x(1-x)}(\beta + (1-\beta)x)} \tag{3-2}$$

With $\beta > 0$ is the invariant measure of the maps $\phi_N^{(i)}(x,\alpha)$ provided that, we choose the parameter α in the following form:

$$\alpha = \frac{\sum_{k=0}^{[\frac{N-1}{2}]} C_{2k+1}^N \beta^{-k}}{\sum_{k=0}^{[\frac{N}{2}]} C_{2k}^N \beta^{-k}} \tag{3-3}$$

in $\phi_N^{(i)}(x,\alpha)$ maps for odd values of $\mathbf{N}$, and;

$$\alpha = \frac{\beta \sum_{k=0}^{[\frac{N}{2}]} C_{2k}^N \beta^{-k}}{\sum_{k=0}^{[\frac{N-1}{2}]} C_{2k+1}^N \beta^{-k}} \tag{3-4}$$

in $\phi_N^{(i)}(x,\alpha)$ maps for even values of $\mathbf{N}$, where the symbol [ ] means greatest integer part. As it is shown in Appendix A, for satisfying the invariant measure $\mu(x,\beta)$ with α, we obtain:

$$\alpha = \frac{B(\frac{1}{\beta})}{A(\frac{1}{\beta})} \tag{3-5}$$

with polynomials $A(x)$ and $B(x)$ defined as:

$$A(x) = \sum_{k=0}^{[\frac{N}{2}]} C_{2k}^N x^k \tag{3-6}$$

$$B(x) = \sum_{k=0}^{[\frac{N-1}{2}]} C_{2k+1}^N x^k \tag{3-7}$$

If an invariant measure can be indecomposed into parts that are invariant, the measure is called non-ergodic. There may be several invariant measure for a dynamical system. If there is a fixed point x^*, then a point distribution $\delta(x - x^*)$ in that point is an invariant measure,

even if the fixed point is unstable. Therefore for the fixed point part of the piecewise map i.e., $y \in [y_0, 1]$ for $\phi_2^{(1)}(x,\alpha)$ and $y \in [0, y_0]$ for both maps $\phi_3^{(1,2)}(x,\alpha)$ and $\phi_2^{(2)}(x,\alpha)$, the average density measure $\mu(x,\beta)$ has the following asymptotic form of the delta function as α goes to the zero and one, respectively;

$$\mu_{av}(x,\alpha) \stackrel{\alpha\to 0}{\longrightarrow} \delta(x)$$

$$\mu_{av}(x,\alpha) \stackrel{\alpha\to 1}{\longrightarrow} \delta(x-1)$$

where the first one corresponds to invariant measure associated with the fixed point at $x = 0$ and the latter one corresponds to fixed point at $x = 1$.

Since for small values of **K** the parameter of the elliptic function, the elliptic chaotic maps of **cn** and **sn** types are topologically conjugate to the one-parameter families of trigonometric chaotic maps, we can obtain the invariant measure of these maps for small **K** [23]. As **K** vanishes, these maps are to trigonometric chaotic maps.

4 Kolmogorov-Sinai Entropy and Lyapunov exponents

Kolmogorov-Sinai (KS) entropy and Lyapunov characteristic exponents are two related ways of measuring 'disorder' in a dynamic system. The definition of them can be found in many textbooks [21]. To calculate KS-entropy here, we use the fact that it is equal to:

$$h(\mu, \phi_N^{(i)}(x,\alpha) = \int \mu(x)dx \ln|\frac{d}{dx}\phi_N^{(i)}(x,\alpha)|$$

which is also a statistical mechanical expression for the Lyapunov characteristic, that is mean divergence rate of two nearby orbits. As it is shown in Appendix B, the KS-entropy of $\phi_N^{(i)}(x,\alpha)$ has following expression:

$$h(\mu, \phi_N^{(i)}(x,\alpha)) = \ln(\frac{N(1+\beta+2\sqrt{\beta})^{N-1}}{(\sum_{k=0}^{[\frac{N}{2}]} C_{2k}^N \beta^k)(\sum_{k=0}^{[\frac{N-1}{2}]} C_{2k+1}^N \beta^k)})$$

A useful numerical way to characterize chaotic phenomena in dynamic systems is by means of the Lyapunov exponents that describe the separation rate of systems whose initial conditions differ by a small perturbation. Suppose that there is a small change $\delta x(0)$ in the initial state $x(0)$. At time **t** this has changed to $\delta x(t)$ given by:

$$\delta x(t) \approx \delta x(0)|\frac{d\phi'}{dx}(x(0))| = \delta x(0)|\phi'(x(t-1)).\phi'(x(t-2)).......\phi'(x(0))|, \qquad (4\text{-}8)$$

Where we have used the chain rule to expand the derivative of ϕ. In the limit of infinitesimal perturbations $\delta x(0)$ and infinite time we get an average exponential amplification, the Lyapunov exponent λ ,

$$\lambda = lim_{t\to\infty}\frac{1}{t}\ln|\frac{\delta x(t)}{\delta x(0)}| = lim_{t\to\infty}\frac{1}{t}\ln|\frac{d\phi'}{dx}(x(0))| = lim_{t\to\infty}\sum_{k=0}^{t-1}\ln|\phi'(x(k))| \qquad (4\text{-}9)$$

5 Ergodicity and non-ergodicity

An probabilistic dynamic system is characterized as ergodic or non-ergodic by its marginal probability distributions. If the distributions have e.g. infinite variances so that a process mean cannot be defined, then the system is non-ergodic. An ergodic system has "convergent" qualities over time, variances are finite and a non-time-dependent process mean is clearly defined. Here a brief description of ergodic theory of chaos [19] is presented: let (Ω, F, μ) be a probability space, Ω is the sample space, i.e.; the space of points, ω designating the elementary outcomes of an experiment. F is the σ-field (or σ-algebra) of events. An event is a set A$\subset\Omega$ which is of interest. the σ-field F is the ensemble of all events, i.e., A$\in$F. Also μ designates a probability measure of F. A transformation T is ergodic, if it has the probability that for almost every ω, the orbit $\{\omega, T\omega, T^2\omega, ...\}$ of ω is a sort of replica of Ω itself. Formally, we shall say that T is ergodic if each invariant set A, i.e.; a set such that T^{-1}(A)=A, is trivial in the sense that it has measure either zero or one. T^{-1}(A)=A $\Rightarrow\mu$(A)=0 or μ(A)=1. The transformation T is ergodic (or indecomposable, or metrically

transitive), if in the Birkhoff theorem, for any integrable, real-value function f, the limit value $\hat{f}$ is constant and we have μ-almost everywhere.

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1} f(T^k\omega)d\mu(\omega) = \int_{\Omega} f(\omega)d\mu(\omega)$$

In this case, the average value of $f(.)$, evaluated along the orbit $T^k\omega$, converges μ-almost everywhere to the mathematical expecting or mean of $f(.)$, evaluated on the space Ω. In other word, for ergodic systems, the time average is equal to the space (or phase) average. One further consideration should be added at this point. The equality of KS-entropy and sum of all positive Lyapunov exponents;

$$h_{KS} = \sum_{\lambda_l>0} \lambda_l$$

indicates that in chaotic region, this map is ergodic as Birkhof ergodic theorem predicts [20]. In other words, when the KS-entropy is smaller than the sum of positive LE, the map has characterization of a non-ergodic behavior.

Also, studied based on invariant measure analysis can be useful for confirming the non-ergodicity behavior of a map. For a non-ergodic system we have:

$$\mu^{-1}(A) = \{x \in [0,1] \quad y = M(x) \quad y \in A\}$$

$$\mu(A) < 1$$

i.e., the invariant measure which is not equal to zero or one, appears to be characteristic of non-ergodic behavior.

6 Results and discussion

In this section we present the results of the analysis numerically for piecewise map. Figure 5, 6, 7 and 8 show the variation of LE and KS-entropy with the parameter α. A positive LE implies that two nearby trajectories exponentially diverge (at last locally). Negative LE

indicate contraction along certain directions, and zero LE indicate that along the relevant directions there is neither expansion nor contraction.

In figure 5, the LE and the corresponding KS-entropy's have been shown for $\phi_2^{(1)}(x, \alpha_1)$ and $\phi_2^{(1)}(x, \alpha_2)$ by some points. A quantity of interest is that the KS-entropy is smaller than the sum of positive LE for piecewise maps. Because of this relation, it is clear that this map has characterization of a non-ergodic behavior.

The invariant measure of this map is equal to 0.696 which is smaller than one, therefore this map behaves non-ergodic. The above analysis are presented for $\phi_2^{(2)}(x, \alpha)$, $\phi_3^{(1,2)}(x, \alpha)$ and the elliptic map $\phi_2^{(2)}(x, \alpha)$ (see figures 5-8).The invariant measure of these maps, respectively are equal to 0.6, 0.314, 0.946 which confirm the non-ergodicity behavior of introduced piecewise maps.

7 Conclusion

A recent attempts in introducing the hierarchy of chaotic maps with their invariant measure [16, 17, 22, 23, 24] allows us to advance in answering to a question how to define non-ergodic maps and what are the condition for non-ergodicity in these types of system.

In this paper we introduce the piecewise maps with their invariant measure. Our numerically calculation shows that values of the KS-entropy are smaller than the sum of positive LE for piecewise maps, therefore these maps behaves non-ergodic. Following, the non-ergodicity behavior, also can be confirmed by the invariant measure which is not equal to zero or one.

8 Appendix A

Similar to the calculation of the invariant measure in our pervious papers [16, 17, 22, 23, 24], we present here it for the piecewise chaotic map. In order to prove that measure (3-2)

satisfied equation (3-1), we consider the conjugate map;

$$\tilde{\phi}_N^{(1)}(x,\alpha) = \frac{1}{\alpha^2}\tan^2(N\arctan\sqrt{x}) \tag{8-1}$$

with measure $\tilde{\mu}_{\tilde{\phi}_N}$ related to the measure μ_{ϕ_N} with the following relation:

$$\tilde{\mu}_{\tilde{\phi}_N}(x) = \frac{1}{(1+x)^2}\mu_{\phi_N}(\frac{1}{1+x}).$$

Denoting $\tilde{\phi}_N(x,\alpha)$ on the left-hand side of (8-1) by $\mathbf{y}$ and inverting it, we get:

$$x_k = \tan^2(\frac{1}{N}\arctan\sqrt{y\alpha^2} + \frac{k\pi}{N}), \quad k = 1, ...N. \tag{8-2}$$

Then, taking derivative of x_k with respect to $\mathbf{y}$, we obtain

$$|\frac{dx_k}{dy}| = \frac{\alpha}{N}\sqrt{x_k}(1+x_k)\frac{1}{\sqrt{y}(1+\alpha^2 y)}. \tag{8-3}$$

Substituting the above result in equation (3-1), we get:

$$\tilde{\mu}_{\tilde{\phi}_N}(y)\sqrt{y}(1+\alpha^2 y) = \frac{\alpha}{N}\sum_k \sqrt{x_k}(1+x_k)\tilde{\mu}_{\tilde{\phi}_N}(x_k). \tag{8-4}$$

Now, by considering the following ansatz for the invariant measure $\tilde{\mu}_{\tilde{\phi}_N}(y)$:

$$\tilde{\mu}_{\tilde{\phi}_N}(y) = \frac{\sqrt{\beta}}{\sqrt{y}(1+\beta y)}, \tag{8-5}$$

the above equation reduces to:

$$\frac{1+\alpha^2 y}{1+\beta y} = \frac{\alpha}{N}\sum_{k=1}^{N}(\frac{1+x_k}{1+\beta x_k})$$

which can be written as:

$$\frac{1+\alpha^2 y}{1+\beta y} = \frac{\alpha}{\beta} + (\frac{\beta-1}{\beta^2})\frac{\partial}{\partial\beta^{-1}}\left(\ \ln(\prod_{k=1}^{N}(\beta^{-1}+x_k))\ \right). \tag{8-6}$$

To evaluate the second term in the right-hand side of above formulas, we can write the equation in the following form:

$$0 = \alpha^2 y\cos^2(N\arctan\sqrt{x}) - \sin^2(N\arctan\sqrt{x})$$

$$= \frac{(-1)^N}{(1+x)^N}\left(\ \alpha^2 y(\textstyle\sum_{k=0}^{[\frac{N}{2}]} C_{2k}^N (-1)^N x^k)^2 - x(\sum_{k=0}^{[\frac{N-1}{2}]} C_{2k+1}^N (-1)^N x^k)^2\ \right) = \frac{constant}{(1+x)^N}\prod_{k=1}^{N}(x - x_k),$$

where x_k are the roots of equation (8-1) and they are given by formula (8-2). Therefore, we have:

$$\frac{\partial}{\partial\beta^{-1}}\ln\left(\ \textstyle\prod_{k=1}^{N}(\beta^{-1}+x_k)\ \right) =$$

$$\frac{\partial}{\partial\beta^{-1}}\ln\left(\ (1-\beta^{-1})^N(\alpha^2 y\cos^2(N\arctan\sqrt{-\beta^{-1}}) - \sin^2(N\arctan\sqrt{-\beta^{-1}})\ \right) =$$

$$-\frac{N\beta}{\beta-1} + \frac{\beta N(1+\alpha^2 y)A(1/\beta)}{(A(1/\beta))^2\beta^2 y + (B(1/\beta))^2}. \tag{8-7}$$

In deriving the above formula, we have used the following identities:

$$\cos(N\arctan\sqrt{x}) = \frac{A(-x)}{(1+x)^{N/2}}, \qquad \sin(N\arctan\sqrt{x}) = \sqrt{x}\frac{B(-x)}{(1+x)^{N/2}}. \tag{8-8}$$

inserting the result (8-7) in (8-6), we get:

$$\frac{1+\alpha^2 y}{1+\beta y} = \frac{1+\alpha^2 y}{B(1/\beta)/A(1/\beta) + \beta(\alpha A(1/\beta)/B(1/\beta))y}$$

Hence, to get the final result, we have to choose the parameter α as:

$$\alpha = \frac{B(1/\beta)}{A(1/\beta)}.$$

9 Appendix B

The KS entropy of one-parameter families of chaotic map is given by:

$$h(\mu,\phi(x,\alpha)) = \int \mu(x)dx\ln|\frac{d}{dx}\phi(x,\alpha)|$$

where

$$\varphi(x,\alpha) = y = \frac{1}{\alpha^2}(tan^2(N\arctan\sqrt{x}))$$

Therefore to calculate $h(\mu,\varphi(x,\alpha))$ we have:

$$h(\mu,\varphi(x,\alpha)) = \int_0^\infty \tilde{\mu}(x)dx\ln\left(\left|\frac{N}{\alpha^2}\frac{1}{\sqrt{x}(1+x)}\frac{\sin N(\arctan\sqrt{x})}{\cos^3 N(\arctan\sqrt{x})}\right|\right)$$

using the relation (8-8), we get:

$$h(\mu,\varphi(x,\alpha)) = \frac{1}{\pi}\int_0^\infty \frac{\sqrt{\beta}dx}{\sqrt{x}(1+\beta x)} \ln\left(\left|\frac{N}{\alpha^2}\frac{(1+x)^{N-1}B(-x)}{(A(-x))^3}\right|\right) \tag{9-1}$$

We see that polynomials appearing in the numerator (denominator) of the integrand appearing on the right-hand side of equation (9-1), have $\frac{1}{2}[N-1](\frac{1}{2}[N])$ simple roots, denoted by $x_k^B,\ k=1,...,\frac{1}{2}[N-1](x_k^A,\ k=1,...,\frac{1}{2}[N])$ in the interval $[0,\infty)$. Hence we can writ the above formula in the following form:

$$h(\mu,\varphi(x,\alpha)) = \frac{1}{\pi}\int_0^\infty \frac{\sqrt{\beta}dx}{\sqrt{x}(1+\beta x)} \ln\left(\frac{N}{\alpha^2}\frac{(1+x)^{N-1}\prod_{k=1}^{[(N-1)/2]}|x-x_k^B|}{\prod_{k=1}^{[N/2]}|x-x_k^A|}\right)$$

Now making the following change of variable$\sqrt{\beta}x = \tan\theta$, and taking into account that degree of numerators and denominator are equal for both even and odd values on **N**we get:

$$h(\mu,\varphi(x,\alpha)) = \frac{1}{\pi}\int_0^\infty d\theta \quad \{\ln(\frac{N}{\alpha^2}) + (N-1)\ln|\beta+1+(\beta-1)\cos\theta|$$

$$+\Sigma_{k=1}^{[(N-1)/2]}\ln|1-x_k^B\beta+(1+x_k^B\beta)\cos\theta| - 3\Sigma_{k=1}^{[N/2]}\ln|1-x_k^A\beta+(1+x_k^A\beta)\cos\theta|\}$$

Using the following integrals:

$$\frac{1}{\pi}\int_0^\pi \ln|a+b\cos\theta| = \begin{cases} \ln|\frac{a+\sqrt{a^2-b^2}}{2}| & |a|>|b| \\ \ln|\frac{b}{2}| & |a|\le|b| \end{cases}$$

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Figures captions

FIG. 1. plot of $\phi_2^{(1)}(x,\alpha)$ for $\alpha_1 = 3$ and $\alpha_2 = 1$.

FIG. 2. plot of $\phi_2^{(2)}(x,\alpha)$ for $\alpha_1 = 0.25$ and $\alpha_2 = 0.75$.

FIG. 3. plot of $\phi_3^{(1,2)}(x,\alpha)$ for $\alpha_1 = 1.5$ and $\alpha_2 = 0.2$.

FIG. 4. plot of the elliptic map $\phi_2^{(2)}(x,\alpha)$ for $\alpha_1 = 0.5$ and $\alpha_2 = 2.5$.

FIG. 5. The variation of the Lyapunov exponent (dotted curve) and the KS-entropy (solid curve) of the $\phi_2^{(1)}(x,\alpha)$ with the parameter α. $\square$ and $\star$ show the valuates of Lyapunov exponent and KS-entropy, respectively for $\phi_2^{(1)}(x,\alpha_1)$ and $\phi_2^{(1)}(x,\alpha_2)$.

FIG. 6. The variation of the Lyapunov exponent (dotted curve) and the KS-entropy (solid curve) of the $\phi_2^{(2)}(x,\alpha)$ with the parameter α. $\square$ and $\star$ show the valuates of Lyapunov exponent and KS-entropy, respectively for $\phi_2^{(2)}(x,\alpha_1)$ and $\phi_2^{(2)}(x,\alpha_2)$.

FIG. 7. The variation of the Lyapunov exponent (dotted curve) and the KS-entropy (solid curve) of the $\phi_3^{(1,2)}(x,\alpha)$ with the parameter α. $\square$ and $\star$ show the valuates of Lyapunov exponent and KS-entropy, respectively for $\phi_3^{(1,2)}(x,\alpha_1)$ and $\phi_3^{(1,2)}(x,\alpha_2)$.

FIG. 8. The variation of the Lyapunov exponent (solid curve) of the elliptic map $\phi_2^{(2)}(x,\alpha)$ with the parameter α for $K = 0$. $\square$ and $\star$ show the valuates of Lyapunov exponent and KS-entropy, respectively for $\phi_2^{(2)}(x,\alpha_1)$ and $\phi_2^{(2)}(x,\alpha_2)$.